Proceedings of the Jangjeon Mathematical Society
27 (2024), No. 2, pp. 101 – 109

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2024.27.2.101

ON THE ISOMORPHISM PROBLEM FOR CENTRAL EXTENSIONS I

NOUREDDINE SNANOU

Abstract. Let G_2 be a group which acts trivially on an abelian group G_1. As is well known, each perturbed direct product of G_1 and G_2 under a 2-cocycle $\varepsilon \in Z^2(G_2, G_1)$ determines a central extension of G_1 by G_2. The purpose of this paper is to study perturbed direct products of groups and to decide in some cases how the isomorphism of these groups can be decided. Furthermore, we show that the study of the isomorphism of perturbed direct products of an abelian torsion group and a finite group is reduced to the study of the isomorphism of p-subgroups. We characterize such isomorphisms in various situations with some assumptions on the quotient group.

2020 Mathematics Subject Classification. 20J05, 20J06, 20E22.

Keywords and phrases: central extension, G-isomorphic, upper isomorphic, $\mathcal{A}$-isomorphic, c-isomorphic.

1. Introduction

Deciding the isomorphism of two given groups or even classifying all groups in a certain class is one of the most classical and challenging problems in group theory. The classification of finite simple groups is the first step of the Hölder program which gives us a complete list of finite simple groups [1]. A group G that is not simple can be broken into two smaller groups, namely a nontrivial normal subgroup G_1 (the kernel group) and the corresponding quotient group $G_2 \cong G/G_1$. This is equivalent to say that G is an extension of G_1 by G_2. In particular, if G_1 is a central subgroup of G, then we say that G is a central extension of G_1 by G_2. The question of what groups G are extensions of G_1 by G_2 is called the extension problem and this is the second step of the Hölder program. The solution to the extension problem would give us a complete classification of all finite groups. But, it is not easy to solve this problem, and no general theory exists which characterizes all possible extensions at one time. However, for group extensions with abelian kernel, an answer to the extension problem has been given by Hölder and Schreier by using the group cohomology, but it has some considerable disadvantages [10, Theorem 7.34]. In fact, this answer does not allow us to compute the number of non-isomorphic extensions of G_1 by G_2 (the isomorphism problem). Very recently, we study in [12] the isomorphism problem for split extensions. In [11, 13], we characterize the isomorphism problem for non-split abelian extensions. In fact, most of the results of those studies do not concern general isomorphisms, but only those of certain type, namely leaving one of the two factors or even both invariant. The aim of this

Submitted July 22, 2023.

paper is to give a further contribution to this topic. More precisely, we complete the work with the isomorphism problem for central extension in other special cases. We mainly deal with isomorphisms inducing the identity or a commuting automorphism on the quotient group. Further, we show that the study of the isomorphism of central extensions of an abelian torsion group by a finite group is reduced to the study of the isomorphism of p-subgroups. In this direction, we characterize such isomorphisms in various situations with some assumptions on the quotient group.

Let G be a group. As usual we denote by $Z(G)$, G' and $Aut(G)$, respectively, the center, the derived subgroup, and the automorphism group of G. If G is finite, then $\pi(G)$ denotes the set of prime divisors of the order of G.

2. Preliminaries and Properties

Let G_2 be a group which acts trivially on an abelian group G_1. A normalized 2-cocycle of G_2 with coefficients in G_1 is a map $\varepsilon : G_2 \times G_2 \to G_1$ satisfying the following two conditions:

$$\varepsilon(g,1) = \varepsilon(1,g) = 1 \text{ for all } g \in G_2. \tag{1}$$

$$\varepsilon(h,g)\varepsilon(hg,k) = \varepsilon(g,k)\varepsilon(h,gk) \text{ for all } g,h,k \in G_2. \tag{2}$$

The condition given by the equation (1) is called the normalization condition, and the condition given by (2) is referred to as the 2-cocycle condition. The set of normalized 2-cocycles of G_2 with coefficients in G_1 is an abelian group and denoted by $Z^2(G_2,G_1)$. The trivial 2-cocycle is the 2-cocycle c with $c(g,h) = 1$ for all $g,h \in G_2$. Note that the elements of $Z^2(G_2,G_1)$ are known by factor sets in many books, (see for example [3, 8, 10, 14]). The set of all normalized 2-cocycles which are symmetric forms a subgroup of $Z^2(G_2,G_1)$ and denoted by $SZ^2(G_2,G_1)$. A 2-coboundary of G_2 with coefficients in G_1 is a map $\psi : G_2 \times G_2 \to G_1$ satisfying that for all $y,y' \in G_2 : \psi(y,y') = \eta(y)\eta(yy')^{-1}\eta(y')$ for some $\eta : G_2 \to G_1$. The set of 2-coboundaries of G_2 with coefficients in G_1 is a subgroup of $Z^2(G_2,G_1)$ and denoted by $B^2(G_2,G_1)$. The corresponding factor group $H^2(G_2,G_1) = Z^2(G_2,G_1)/B^2(G_2,G_1)$ is called the second cohomology group of G_2 with coefficients in G_1. The elements of $H^2(G_2,G_1)$ are called cohomology classes. The cohomology class of $\varepsilon \in Z^2(G_2,G_1)$ is denoted by $[\varepsilon]$. Two normalized 2-cocycles are said to be cohomologous if they lie in the same cohomology class.

Let $1 \to G_1 \overset{i}{\to} G \overset{j}{\to} G_2 \to 1$ be a short exact sequence of groups, i.e., an extension of a group G_1 by the group $G_2 \cong G/G_1$. If G_1 is a central subgroup of G, then such an extension is called a central extension of G_1 by G_2. We refer to G_1 as the kernel group, and G_2 as the quotient group for the extension. The Schreier's theorem says that the central extensions of G_1 by G_2 are classified by the non-trivial elements of the second cohomology group $H^2(G_2,G_1)$ with coefficients in G_1 [10, Theorem 7.34]. Split extensions correspond to the trivial equivalence class of $H^2(G_2,G_1)$.

Let G_2 be a group which acts trivially on an abelian group G_1. It is well known that each perturbed direct product of G_1 and G_2 under a 2-cocycle ε determines a central extension of G_1 by G_2. Let $\varepsilon \in Z^2(G_2,G_1)$, the perturbed direct product of G_1 and G_2 under ε is defined as the group $G_1 \underset{\varepsilon}{\times} G_2$ with underlying set $G_1 \times G_2$ and operation given by

$$(x, y) \underset{\varepsilon}{\cdot} (x', y') = (xx'\varepsilon(y, y'), yy')$$

for all x, $x' \in G_1$ and y, $y' \in G_2$. The converse is also true, then each central extension G of G_1 by G_2 is isomorphic to a perturbed direct product of G_1 and G_2, namely $G \cong G_1 \underset{\varepsilon}{\times} G_2$ for some $\varepsilon \in Z^2(G_2, G_1)$ [11, Proposition 2.3]. We can easily see that the perturbed direct product $G_1 \underset{\varepsilon}{\times} G_2$ is abelian if and only if G_2 is abelian and $\varepsilon \in SZ^2(G_2, G_1)$. In particular, we have $G_1 \underset{\varepsilon}{\times} G_2 = G_1 \times G_2$ if and only if ε is the trivial 2-cocycle. But, it is possible for a direct product to be isomorphic to a perturbed direct product as Remark 2.2 shows. In particular, suppose that G_1 and G_2 are abelian groups and ε is non-symmetric. So $G_1 \underset{\varepsilon}{\times} G_2$ is non-abelian and then $G_1 \underset{\varepsilon}{\times} G_2 \ncong G_1 \times G_2$. Furthermore, we have

Proposition 2.1. *Let G_1 be a finite abelian group and G_2 a finite group and let $\varepsilon \in Z^2(G_2, G_1)$. If $SZ^2(G_2, G_1) = \{1\}$, then $G_1 \underset{\varepsilon}{\times} G_2 \cong G_1 \times G_2$ if and only if $\varepsilon = 1$.*

Proof. The if direction is clear. Conversely, suppose that $\varepsilon \neq 1$, by using the 2-cocycle condition, we get $[(x, y), (x', y')] = (\varepsilon(y, y')\varepsilon(y', y)^{-1}, [y, y'])$ for all (x, y), $(x', y') \in G_1 \underset{\varepsilon}{\times} G_2$. So, $(G_1 \underset{\varepsilon}{\times} G_2)' \cong H_\varepsilon \underset{\varepsilon'}{\times} G_2'$ such that $\varepsilon' = res_{G_2' \times G_2'}(\varepsilon)$ and H_ε is generated by the elements of the form $\varepsilon(y, y')\varepsilon(y, y')^{-1}$ where y, $y' \in G_2$. By assumption, we have H_ε is a nontrivial subgroup of G_1. But, $(G_1 \times G_2)' = G_2'$ which implies that $G_1 \underset{\varepsilon}{\times} G_2 \ncong G_1 \times G_2$, as required. □

Remark 2.2. *Let $\varepsilon \in Z^2(G_2, G_1)$ and $G = G_1 \underset{\varepsilon}{\times} G_2$ be a finite group. Under some conditions on G_1 and G_2, the group G can also be decomposed as a direct product of G_1 and G_2. Indeed, by* [7, Proposition 2.1.7], *$G' \cap G_1$ is isomorphic to a subgroup of the Schur multiplier $M(G/G_1)$ of G/G_1. So if $|G_1|$ and $|M(G_2)|$ are coprime, then $G' \cap G_1 = 1$. Further, if G_2 is perfect, then so is G/G_1. Hence $G/G_1 = G'G_1/G_1$, which implies that $G = G'G_1$. Thus, $G = G' \times G_1$ and then $G \cong G_1 \times G_2$.*

Now, in view of the preceding discussion, the following problem seems natural.

Problem 2.3. *Find necessary and sufficient conditions on ε_1 and ε_2 under which the central extensions $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are isomorphic.*

To begin, let $\varepsilon_1, \varepsilon_2 \in Z^2(G_2, G_1)$ and φ a group homomorphism from $G_1 \underset{\varepsilon_1}{\times} G_2$ to $G_1 \underset{\varepsilon_2}{\times} G_2$. Let $pr_i : G_1 \underset{\varepsilon_2}{\times} G_2 \to G_i$ be the ith canonical projection and $t_i : G_i \to G_1 \underset{\varepsilon_1}{\times} G_2$ be the ith canonical injection. Set $\varphi_{ij} = pr_i \circ \varphi \circ t_j$, where $1 \leq i, j \leq 2$. So we can write φ in the matrix form: $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$. Furthermore, we have the following lemma which we need in the sequel.

Lemma 2.4. [11, Lemma 3.1] *Let* $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ *be a group homomorphism from* $G_1 \underset{\varepsilon_1}{\times} G_2$ *to* $G_1 \underset{\varepsilon_2}{\times} G_2$. *Then*

$$\varphi(x,y) = (\varphi_{11}(x)\varphi_{12}(y)\varepsilon_2(\varphi_{21}(x), \varphi_{22}(y)),\ \varphi_{21}(x)\varphi_{22}(y)) \tag{3}$$

for all $x \in G_1$, *and* $y \in G_2$.

3. Isomorphisms inducing the identity on the quotient group

Definition 3.1. *The perturbed direct products* $G_1 \underset{\varepsilon_1}{\times} G_2$ *and* $G_1 \underset{\varepsilon_2}{\times} G_2$ *are called* G_2*-isomorphic if there exists an isomorphism* $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ *between them such that* $\varphi_{22} = id_{G_2}$.

In the following, we give an interesting result for a special class of nonnilpotent quotient groups, namely for those that have trivial center.

Proposition 3.2. *Let* G_2 *be a centerless group which acts trivially on an abelian group* G_1. *The perturbed direct products* $G_1 \underset{\varepsilon_1}{\times} G_2$ *and* $G_1 \underset{\varepsilon_2}{\times} G_2$ *are* G_2*-isomorphic if and only if there exists* $\sigma \in Aut(G_1)$ *such that* $(\sigma \circ \varepsilon_1)\varepsilon_2^{-1} \in B^2(G_2, G_1)$.

Proof. Suppose that the perturbed direct products $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are isomorphic by an isomorphism $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & id_{G_2} \end{pmatrix}$. Since pr_2 and t_1 are group homomorphisms, so is φ_{21}. Furthermore, we see that $\varphi(x,1) \underset{\varepsilon_2}{\bullet} \varphi(1,y) = \varphi(1,y) \underset{\varepsilon_2}{\bullet} \varphi(x,1)$. So by applying formula (3), we get $\varphi_{21}(x)y = y\varphi_{21}(x)$ for all $x \in G_1$, and $y \in G_2$. Thus, we have $\varphi_{21} \in Hom(G_1, Z(G_2))$. Since G_2 is centerless, it follows that $\varphi_{21} = 1$. Hence, by [13, Theorem 3.7], there exists $\sigma = \varphi_{11} \in Aut(G_1)$ such that $(\sigma \circ \varepsilon_1)\varepsilon_2^{-1} \in B^2(G_2, G_1)$. Conversely, since $(\sigma \circ \varepsilon_1)\varepsilon_2^{-1} \in B^2(G_2, G_1)$, it follows that there exists a map $\eta : G_2 \to G_1$ such that $((\sigma \circ \varepsilon_1)\varepsilon_2^{-1})(y, y') = \eta(y)\eta(y')\eta(yy')^{-1}$ for all $y,\ y' \in G_2$. By the normalization condition, we have $\eta(1) = 1$. So, the bijection φ defined by $\varphi(x,y) = (\sigma(x)\eta(y),\ y)$ is clearly an isomorphism. As required. □

Let G_1 be an abelian torsion group, i.e. all elements of G_1 are of finite order. Then G_1 is a restricted direct product of all p-components G_{1p}, where p runs through the set of prime numbers. Let G_2 be a finite group which acts trivially on G_1 and $\varepsilon \in Z^2(G_2, G_1)$. Let $\pi(G_2) = \{p_1, p_2, \ldots, p_k\}$ and G_{2i} be a Sylow p_i-subgroup of G_2 for each $1 \leq i \leq k$. Clearly, we have $\varepsilon_i = res_{G_{2i}}(\varepsilon) \in Z^2(G_{2i}, G_1)$. So, in view of the previous preposition, we get the following result.

Theorem 3.3. *Keep the preceding notations and assumptions and suppose that* G_2 *is centerless. The perturbed direct products* $G_1 \underset{\varepsilon_1}{\times} G_2$ *and* $G_1 \underset{\varepsilon_2}{\times} G_2$ *are* G_2*-isomorphic if and only if* $G_{1p_i} \underset{\varepsilon_{1i}}{\times} G_{2i}$ *and* $G_{1p_i} \underset{\varepsilon_{2i}}{\times} G_{2i}$ *are* G_{2i}*-isomorphic for all* $1 \leq i \leq k$.

Proof. Suppose that the groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are G_2-isomorphic. By Proposition 3.2, there exists $\sigma \in Aut(G_1)$ such that $(\sigma \circ \varepsilon_1)\varepsilon_2^{-1} \in B^2(G_2, G_1)$. So $res_{G_{2i}}((\sigma \circ \varepsilon_1)\varepsilon_2^{-1}) \in B^2(G_{2i}, G_1)$. Since G_{2i} is a p_i-group and G_1 is torsion, by [6, Lemma 1.5], it follows that $H^2(G_{2i}, G_1) = H^2(G_{2i}, G_{1p_i})$. Further, we have $\sigma_i = res_{G_{1p_i}}(\varphi_{11}) \in Aut(G_{1p_i})$. So $(\sigma_i \circ \varepsilon_{1i})\varepsilon_{2i}^{-1} \in B^2(G_{2i}, G_{1p_i})$, which implies necessity. Conversely, suppose that $G_{1p_i} \underset{\varepsilon_{1i}}{\times} G_{2i}$ and $G_{1p_i} \underset{\varepsilon_{2i}}{\times} G_{2i}$ are G_{2i}-isomorphic for all $1 \leq i \leq k$. So by Proposition 3.2, there exists $\sigma_i \in Aut(G_{1p_i})$ such that $(\sigma_i \circ \varepsilon_{1i})\varepsilon_{2i}^{-1} \in B^2(G_{2i}, G_{1p_i})$ for each $1 \leq i \leq k$. So $\sigma = (\sigma_i)_{1 \leq i \leq k} \in Aut(G_1)$ and then $res_{G_{2i}}([(\sigma \circ \varepsilon_1)\varepsilon_2^{-1}]) = 1$ in $H^2(G_{2i}, G_{1p_i})$. Apply the corestriction map $cores_{G_{2i}} : H^2(G_{2i}, G_{1p_i}) \to H^2(G_2, G_{1p_i})$. Then by using [15, Corollary 2.4.9], we get $[(\sigma \circ \varepsilon_1)\varepsilon_2^{-1}]^{|G_2:G_{2i}|} = 1$ for all $1 \leq i \leq k$. Hence, the order of $[(\sigma \circ \varepsilon_1)\varepsilon_2^{-1}]$ is coprime with all elements of $\pi(G_2)$. But, by [15, Proposition 3.1.6], we have $H^2(G_2, G_1)^{|G_2|} = 1$, which implies that $[(\sigma \circ \varepsilon_1)\varepsilon_2^{-1}] = 1$ in $H^2(G_2, G_1)$, as required. □

Definition 3.4. *Let H be a subgroup of G_1. The perturbed direct products $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are called (H, G_2)-isomorphic if there exists a G_2-isomorphism* $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & id_{G_2} \end{pmatrix}$ *between them such that $\varphi_{11}/H = id_H$.*

Proposition 3.5. *Let G_2 be a group which acts trivially on an abelian group G_1. Let $H = Im(\varepsilon_1)$. The perturbed direct products $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are (H, G_2)-isomorphic if and only if $\varepsilon_1\varepsilon_2^{-1} \in B^2(G_2, G_1)$.*

Proof. Indeed, if the perturbed direct products $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are (H, G_2)-isomorphic, then there exists an isomorphism $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & id_{G_2} \end{pmatrix}$ such that $\varphi_{21} \in Hom(G_1, Z(G_2))$ and $\varphi_{11}/H = id_H$. So, evaluate the left hand side and right hand side of the equality $\varphi(1, y) \underset{\varepsilon_2}{\bullet} \varphi(1, y') = \varphi(\varepsilon_1(y, y'), yy')$, we obtain

(1) $\varphi_{21}(\varepsilon_1(y, y'))yy' = yy'$,

(2) $\varphi_{11}(\varepsilon_1(y, y'))\varphi_{12}(yy')\varepsilon_2(\varphi_{21}(\varepsilon_1(y, y')), yy') = \varphi_{12}(y)\varphi_{12}(y')\varepsilon_2(y, y')$.

The first equality implies that $Im(\varepsilon_1) \leq Ker(\varphi_{21})$. So the second equality gives us

$$\varphi_{11}(\varepsilon_1(y, y'))\varepsilon_2(y, y')^{-1} = \varphi_{12}(y)\varphi_{12}(y')\varphi_{12}(yy')^{-1}.$$

Thus $(\varphi_{11} \circ \varepsilon_1)\varepsilon_2^{-1} \in B^2(G_2, G_1)$ and therefore $\varepsilon_1\varepsilon_2^{-1} \in B^2(G_2, G_1)$ since $\varphi_{11}/H = id_H$. The proof of the converse is clear and similar to the proof of the converse of Proposition 3.2 and then it is omitted. □

Using the preceding proposition, the proof of the following result is similar to the proof of Theorem 3.3 and then we omit the details.

Theorem 3.6. *Let G_2 be a finite group which acts trivially on an abelian torsion group G_1. Let $H = Im(\varepsilon_1)$ and $H_i = Im(\varepsilon_{1i})$ for all $1 \leq i \leq k$. The perturbed direct products $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are (H, G_2)-isomorphic* *if and only if* $G_{1p_i} \underset{\varepsilon_{1i}}{\times} G_{2i}$ *and* $G_{1p_i} \underset{\varepsilon_{2i}}{\times} G_{2i}$ *are* (H_i, G_{2i})*-isomorphic for all* $1 \leq i \leq k$.

4. Isomorphisms leaving the kernel group invariant

We need the following definition.

Definition 4.1. *Let* $\varepsilon_1, \varepsilon_2 \in Z^2(G_2, G_1)$. *The perturbed direct products* $G_1 \underset{\varepsilon_1}{\times} G_2$ *and* $G_1 \underset{\varepsilon_2}{\times} G_2$ *are called upper isomorphic if there exists an isomorphism* $\varphi : G_1 \underset{\varepsilon_1}{\times} G_2 \longrightarrow G_1 \underset{\varepsilon_2}{\times} G_2$ *leaving* G_1 *invariant.*

The following lemma can be viewed as an immediate consequence of [13, Theorem 3.7].

Lemma 4.2. *The perturbed direct products* $G_1 \underset{\varepsilon_1}{\times} G_2$ *and* $G_1 \underset{\varepsilon_2}{\times} G_2$ *are isomorphic by an isomorphism* $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ 1 & \varphi_{22} \end{pmatrix}$ *if and only if* $\varphi_{11} \in Aut(G_1)$ *and* $\varphi_{22} \in Aut(G_2)$ *such that*

$$(\varphi_{11} \circ \varepsilon_1)(\varepsilon_2^{-1} \circ (\varphi_{22} \times \varphi_{22})) \in B^2(G_2, G_1).$$

With the notations and assumptions that preceded Theorem 3.3, we have the following.

Proposition 4.3. *Let* G_2 *be a cyclic group which acts trivially on an abelian torsion group* G_1. *The perturbed direct products* $G_1 \underset{\varepsilon_1}{\times} G_2$ *and* $G_1 \underset{\varepsilon_2}{\times} G_2$ *are upper isomorphic if and only if* $G_{1p_i} \underset{\varepsilon_{1i}}{\times} G_{2i}$ *and* $G_{1p_i} \underset{\varepsilon_{2i}}{\times} G_{2i}$ *are upper isomorphic for all* $1 \leq i \leq k$.

Proof. Indeed, by [13, Proposition 3.11], the groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are upper isomorphic if and only if there exists $\sigma \in Aut(G_1)$ such that $(\sigma \circ \varepsilon_1)\varepsilon_2^{-1} \in B^2(G_2, G_1)$. Thus, using the same arguments as those used in the proof of Theorem 3.3, we get the required result. □

If G is a group and $g \in G$, we will write γ_g for the inner automorphism determined by g, i.e. γ_g maps an element x to gxg^{-1}.

Theorem 4.4. *Let* G_2 *be a group which acts trivially on an abelian torsion group* G_1. *If the perturbed direct products* $G_1 \underset{\varepsilon_1}{\times} G_2$ *and* $G_1 \underset{\varepsilon_2}{\times} G_2$ *are upper isomorphic, then* $G_{1p_i} \underset{\varepsilon_{1i}}{\times} G_{2i}$ *and* $G_{1p_i} \underset{\varepsilon_{2i}}{\times} G_{2i}$ *are upper isomorphic for all* $1 \leq i \leq k$.

Proof. Indeed, if the groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are upper isomorphic, then by Lemma 4.2, there exists an isomorphism $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ 1 & \varphi_{22} \end{pmatrix}$ between them such that $\varphi_{11} \in Aut(G_1)$ and $\varphi_{22} \in Aut(G_2)$. Since $G_{2i} \in Syl_{p_i}(G_2)$, it follows that $\varphi_{22}(G_{2i}) = g_i G_{2i} g_i^{-1}$ for some $g_i \in G_2$ and then $(\gamma_{g_i^{-1}} \circ \varphi_{22})(G_{2i}) = G_{2i}$. Therefore $\rho_i = res_{G_{2i}}(\gamma_{g_i^{-1}} \circ \varphi_{22}) \in Aut(G_{2i})$. Further, we have $\sigma_i = res_{G_{1p_i}}(\varphi_{11}) \in Aut(G_{1p_i})$ since G_1 is torsion. Now, by a simple calculation, we get $\gamma_{t_2(g_i)^{-1}} \circ \varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12}^i \\ 1 & \varphi_{22}^i \end{pmatrix}$, where $\varphi_{22}^i = \gamma_{g_i^{-1}} \circ \varphi_{22}$ and $\varphi_{12}^i(y) = \varphi_{12}(y)\varepsilon_2(g_i^{-1}, g_i)^{-1}\varepsilon_2(g_i^{-1}, \varphi_{22}(y))\varepsilon_2(g_i^{-1}\varphi_{22}(y), g_i)$ for all $y \in G_2$. Hence, the groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are lower isomorphic by the isomorphism $\varphi' = \gamma_{t_2(g_i)^{-1}} \circ \varphi$. Therefore, by Lemma 4.2, we have $(\varphi_{11} \circ \varepsilon_1)(\varepsilon_2^{-1} \circ (\varphi_{22}^i \times \varphi_{22}^i)) \in B^2(G_2, G_1)$. Thus, by applying the restriction map $res_{G_{2i}}$, we get $(\sigma_i \circ \varepsilon_{1i})(\varepsilon_{2i}^{-1} \circ (\rho_i \times \rho_i)) \in B^2(G_{2i}, G_{1p_i})$. This completes the proof. □

In particular, if G_2 is a finite nilpotent group, then the converse of the preceding result holds, as shown in the following proposition.

Proposition 4.5. *Let G_2 be a finite nilpotent group which acts trivially on an abelian torsion group G_1. Suppose that $G_{1p_i} \underset{\varepsilon_{1i}}{\times} G_{2i}$ and $G_{1p_i} \underset{\varepsilon_{2i}}{\times} G_{2i}$ are upper isomorphic for all $1 \leq i \leq k$. Then, the perturbed direct products $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are upper isomorphic.*

Proof. Indeed, if $G_{1p_i} \underset{\varepsilon_{1i}}{\times} G_{2i}$ and $G_{1p_i} \underset{\varepsilon_{2i}}{\times} G_{2i}$ are upper isomorphic, then by Lemma 4.2, there exist $\sigma_i \in Aut(G_{1p_i})$, $\rho_i \in Aut(G_{2i})$ such that $(\sigma_i \circ \varepsilon_{1i})(\varepsilon_{2i}^{-1} \circ (\rho_i \times \rho_i)) \in B^2(G_{2i}, G_{1p_i})$. Since G_2 is nilpotent, it is the direct product of its Sylow subgroups and then $\rho = (\rho_i)_{1 \leq i \leq k} \in Aut(G_2)$. Furthermore, we have $\sigma = (\sigma_i)_{1 \leq i \leq k} \in Aut(G_1)$. Thus, we have $res_{G_{2i}}[(\sigma \circ \varepsilon_1)(\varepsilon_2^{-1} \circ (\rho \times \rho))] = 1$ in $H^2(G_{2i}, G_{1p_i})$. The rest of the proof is similar to the second part of the proof of Theorem 3.3, and so is omitted. □

5. Isomorphisms inducing a commuting automorphism on the quotient group

Let G be a group. An automorphism ρ of G is called a commuting automorphism of G if for each $x \in G$, $\rho(x)$ commutes with x. The set of all commuting automorphisms of G is denoted by $\mathcal{A}(G)$. The group $Aut_c(G) = C_{Aut(G)}(G/Z(G))$ of central automorphisms of G is always a subset of $\mathcal{A}(G)$.

Definition 5.1. *The perturbed direct products $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are called $\mathcal{A}$-isomorphic if there exists an isomorphism $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$ between them such that $\varphi_{22} \in \mathcal{A}(G_2)$. In particular, if $\varphi_{22} \in Aut_c(G_2)$, then the groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are called c-isomorphic.*

Proposition 5.2. *Let G_2 be a centerless perfect group. Then, the perturbed direct products $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are $\mathcal{A}$-isomorphic if and only if there exists $\sigma \in Aut(G_1)$ such that $(\sigma \circ \varepsilon_1)\varepsilon_2^{-1} \in B^2(G_2, G_1)$.*

Proof. Since G_2 is a centerless perfect group, it follows that $\mathcal{A}(G_2) = \{id_{G_2}\}$ [9]. Therefore, the result follows directly from Proposition 3.2. □

Let G_2 be a finite group such that $\pi(G_2) = \{p_1, p_2, \ldots, p_k\}$. Let G_{2i} be a Sylow p_i-subgroup of G_2 for each $1 \leq i \leq k$. Then, we have the following result.

Proposition 5.3. *Suppose that all of the sylow subgroups of G_2 are of maximal class such that $\log_{p_i}|G_{2i}| \geq 4$ for all $1 \leq i \leq k$. Then, the perturbed direct products $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are upper $\mathcal{A}$-isomorphic if and only if they are upper c-isomorphic.*

Proof. Assume that $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are $\mathcal{A}$-isomorphic by an isomorphism $\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ 1 & \varphi_{22} \end{pmatrix}$. By [4, Remark 4.2 (ii)], each Sylow subgroup of G_2 is normalized by $\mathcal{A}(G_2)$. Therefore, we have $Res_{G_{2i}}(\varphi_{22}) \in \mathcal{A}(G_{2i})$ for all $1 \leq i \leq k$. Using the assumptions and [5, Theorem 3.4], we get $\mathcal{A}(G_{2i}) = Aut_c(G_{2i})$ for all $1 \leq i \leq k$. Hence, by [4, Remark 4.3], we have $\varphi_{22} \in Aut_c(G_2)$. Thus, $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are upper c-isomorphic. The other direction is clear since $Aut_c(G_2)$ is a subset of $\mathcal{A}(G_2)$. □

With the notations that preceded Theorem 3.3, we obtain the following proposition.

Proposition 5.4. *Let G_2 be a finite nilpotent group which acts trivially on an abelian torsion group G_1. If all the sylow subgroups of G_2 are of coclass at most two, then the perturbed direct products $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are upper $\mathcal{A}$-isomorphic if and only if $G_{1p_i} \underset{\varepsilon_{1i}}{\times} G_{2i}$ and $G_{1p_i} \underset{\varepsilon_{2i}}{\times} G_{2i}$ are upper $\mathcal{A}$-isomorphic for all $1 \leq i \leq k$.*

Proof. Indeed, by Lemma 4.2, if the groups $G_{1p_i} \underset{\varepsilon_{1i}}{\times} G_{2i}$ and $G_{1p_i} \underset{\varepsilon_{2i}}{\times} G_{2i}$ are upper $\mathcal{A}$-isomorphic, then there exist $\sigma \in Aut(G_1)$, $\rho \in \mathcal{A}(G_2)$ such that $(\sigma \circ \varepsilon_1)(\varepsilon_2^{-1} \circ (\rho \times \rho)) \in B^2(G_2, G_1)$. Clearly, we have $\rho_i = res_{G_{2i}}(\rho) \in \mathcal{A}(G_{2i})$ and $\sigma_i = res_{G_{1p_i}}(\sigma) \in Aut(G_{1p_i})$ for all $1 \leq i \leq k$. Hence, $res_{G_{2i}}((\sigma \circ \varepsilon_1)(\varepsilon_2^{-1} \circ (\rho \times \rho))) = (\sigma_i \circ \varepsilon_{1i})(\varepsilon_{2i}^{-1} \circ (\rho_i \times \rho_i)) \in B^2(G_{2i}, G_{1p_i})$. This proves the only if direction. For the converse, suppose that the groups $G_{1p_i} \underset{\varepsilon_{1i}}{\times} G_{2i}$ and $G_{1p_i} \underset{\varepsilon_{2i}}{\times} G_{2i}$ are upper $\mathcal{A}$-isomorphic for all $1 \leq i \leq k$. By Proposition 4.5, the groups $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are isomorphic by an isomorphism $\varphi = \begin{pmatrix} \sigma & \eta \\ 1 & \rho \end{pmatrix}$ such that $\rho = (\rho_i)_{1 \leq i \leq k} \in Aut(G_2)$ where $\rho_i \in \mathcal{A}(G_{2i})$. By [2, Corollary 3.4], we have $\mathcal{A}(G_2) \cong \prod_{i=1}^{k} \mathcal{A}(G_{2i})$. Thus $\rho \in \mathcal{A}(G_2)$ and then $G_1 \underset{\varepsilon_1}{\times} G_2$ and $G_1 \underset{\varepsilon_2}{\times} G_2$ are upper $\mathcal{A}$-isomorphic. □

Acknowledgements

I am thankful to the editor Hwankoo Kim for handling this manuscript, and to the referee for suggesting some minor changes and helpful corrections.

References

[1] M. Aschbacher, *The status of the classification of the finite simple groups*, Notices Amer. Math. Soc. 51(7) (2004), 736–740.

[2] N. Azimi Shahrabi, M. Akhavan-Malayeri, and F. Vosooghpour, *Commuting automorphisms of finite 2-groups of almost maximal class I*, J. Algebra Appl. 18(11) (2019), 1950208.

[3] K. S. Brown, *Cohomology of Groups*, Springer GTM, Vol. 87, Springer-Verlag, New York, 1982.
[4] M. Deaconescu, G. Silberberg, G.L. Walls, *On commuting automorphisms of groups*, Arch. Math. 79 (2002), 423–429.
[5] S. Fouladi and R. Orfi, *Commuting automorphisms of some finite groups*, Glas. Mat. Ser. III. 48(1) (2013), 91–96.
[6] G. Karpilovsky, *Group Representations*, Vol. 2, North-Holland, Amsterdam, London, New York, Tokyo, 1993.
[7] G. Karpilovsky, *The Schur Multipliers*, Oxford Science Publications, London Mathematical society Monographs, New Series 2, Clarendon Press, Oxford, 1987.
[8] S. MacLane, *Homology*, Springer Grundlehren, Vol. 114, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.
[9] T. J. Laffey, *Problems and solutions: solutions of elementary problems: E3039*, Amer. Math. Monthly 93(10) (1986), 816–817.
[10] J. J. Rotman, *An Introduction to the Theory of Groups*, Springer GTM, Vol. 148, 4th ed., Springer-Verlag, New York, 1995.
[11] N. Snanou and M. E. Charkani, *On non-split abelian extensions*, Bull. Iran. Math. Soc., 47(3) (2021), 743–753.
[12] N. Snanou and M. E. Charkani, *On the isomorphism problem for split extensions*, J. Algebra Appl. 23(2) (2024), 2430002.
[13] N. Snanou, *On non-split abelian extensions II*, Asian-Eur. J. Math. 14(9) (2021), 2150164.
[14] C. Weibel, *An Introduction to Homological Algebra*, Cambridge studies in advanced mathematics, Vol. 38, Cambridge University Press, Cambridge, 1994.
[15] E. Weiss, *Cohomology of Groups*, Academic Press, New York, 1969.

Department of Mathematics, Faculty of Sciences Dhar El Mahraz, Sidi Mohamed Ben Abdellah University, Fez, Morocco
Email address: `noureddine.snanou@usmba.ac.ma`